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The topology of the space of rational curves on a toric variety and related problems

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Abstract

We report about the recent joint work with A. Kozłowski [16] (cf. [15], [18]) concerning to the topology of spaces of rational curves on a toric variety and related problems.

1 Introduction

Spaces of maps. For connected spaces X and Y , let $\text{Map}^*(X, Y)$ denote the space consisting of all base point preserving continuous maps from X to Y with the compact-open topology, and for each class $D \in \pi_0(\text{Map}^*(X, Y))$, let $\text{Map}_D^*(X, Y)$ denote the path-component of $\text{Map}^*(X, Y)$ corresponding to the homotopy class D . When $X = \mathbb{CP}^1$ and Y is a complex manifolds, we denote by $\text{Hol}_D^*(S^2, Y)$ the space of all base point preserving holomorphic maps $f \in \text{Map}_D^*(S^2, Y) = \Omega_D^2 Y$.

Convex rational polyhedral cones. A *convex rational polyhedral cone* is the subset of \mathbb{R}^m of the form

$$(1.1) \quad \sigma = \text{Cone}(S) = \text{Cone}(\mathbf{m}_1, \dots, \mathbf{m}_s) = \left\{ \sum_{k=1}^s \lambda_k \mathbf{m}_k : \lambda_k \geq 0 \right\}$$

for some finite set $S = \{\mathbf{m}_k : 1 \leq k \leq s\} \subset \mathbb{Z}^m$ and it is called *strongly convex* if $\sigma \cap (-\sigma) = \{\mathbf{0}_m\}$, where we set $\mathbf{0}_m = (0, 0, \dots, 0) \in \mathbb{R}^m$. When S is the emptyset \emptyset , we set $\text{Cone}(\emptyset) = \{\mathbf{0}_m\}$ and we may also regard it as one of convex rational polyhedral cones.

Fans and toric varieties. Let X be an m dimensional irreducible normal algebraic variety over \mathbb{C} . One says that X is a *toric variety* if it has an algebraic action of of an algebraic torus $\mathbb{T}_{\mathbb{C}}^m = (\mathbb{C}^*)^m$, such that the orbit $\mathbb{T}_{\mathbb{C}}^m \cdot *$ of some point $*$ in X is dense in X and isomorphic to $\mathbb{T}_{\mathbb{C}}^m$. A toric variety X is characterized up to isomorphism by its *fan* Σ , which is a finite collection of strongly convex rational polyhedral cones in \mathbb{R}^m such that every face τ of $\sigma \in \Sigma$ belongs to Σ and the intersection $\sigma_1 \cap \sigma_2$ of any two elements $\sigma_1, \sigma_2 \in \Sigma$ is a face of each σ_k ($k = 1, 2$). We denote by X_{Σ} the toric variety associated to the fan Σ .

Polyhedral products and homogenous coordinates. Let K be a simplicial complex on the index set $[r] = \{1, 2, \dots, r\}$,¹ and let (X, A) be pair of spaces such that $A \subset X$. Then define the *polyhedral product* $\mathcal{Z}_K(X, A)$ with respect to K by the union $\mathcal{Z}_K(X, A) = \bigcup_{\sigma \in K} (X, A)^\sigma$, where $(X, A)^\sigma = \{(x_1, \dots, x_r) \in X^r : x_k \in A \text{ if } k \notin \sigma\}$.

Note that the space $\mathcal{Z}_K(D^2, S^1)$ is usually called the *moment-angle complex* of K .

Definition 1.1. Let Σ be a fan in \mathbb{R}^m such that $\{\mathbf{0}_m\} \subsetneq \Sigma$, and let $\Sigma(1) = \{\rho_1, \dots, \rho_r\}$ denote the set of all one dimensional cones in Σ .

(i) For each integer $1 \leq k \leq r$, we denote by $\mathbf{n}_k \in \mathbb{Z}^m$ the *primitive generator* of ρ_k , such that $\rho_k \cap \mathbb{Z}^m = \mathbb{Z}_{\geq 0} \cdot \mathbf{n}_k$. Note that $\rho_k = \text{Cone}(\mathbf{n}_k)$.

(ii) Let \mathcal{K}_Σ denote the *underlying simplicial complex* of Σ defined by

$$(1.2) \quad \mathcal{K}_\Sigma = \left\{ \{i_1, \dots, i_s\} \subset [r] : \text{Cone}(\mathbf{n}_{i_1}, \dots, \mathbf{n}_{i_s}) \in \Sigma \right\}.$$

It is easy to see that \mathcal{K}_Σ is a simplicial complex on the index set $[r]$.

(iii) Next, define the subgroup $G_\Sigma \subset \mathbb{T}_{\mathbb{C}}^r$ by

$$(1.3) \quad G_\Sigma = \left\{ (\mu_1, \dots, \mu_r) \in \mathbb{T}_{\mathbb{C}}^r : \prod_{k=1}^r (\mu_k)^{\langle \mathbf{n}_k, \mathbf{m} \rangle} = 1 \text{ for all } \mathbf{m} \in \mathbb{Z}^m \right\},$$

where $\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{k=1}^m u_k v_k$ for $\mathbf{u} = (u_1, \dots, u_m)$ and $\mathbf{v} = (v_1, \dots, v_m) \in \mathbb{R}^m$.

(iv) Consider the natural G_Σ -action on $\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}, \mathbb{C}^*)$ given by coordinate-wise multiplication, i.e.

$$(\mu_1, \dots, \mu_r) \cdot (x_1, \dots, x_r) = (\mu_1 x_1, \dots, \mu_r x_r)$$

for $((\mu_1, \dots, \mu_r), (x_1, \dots, x_r)) \in G_\Sigma \times \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}, \mathbb{C}^*)$. □

Let $I(\mathcal{K}_\Sigma) = \{\sigma \subset [r] : \sigma \notin \mathcal{K}_\Sigma\}$ and consider the orbit space $\mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}, \mathbb{C}^*)/G_\Sigma$. Now recall the following result due to D. Cox.

Theorem 1.2 (D. Cox; [4], [5]). *Let Σ be a fan in \mathbb{R}^m as in Definition 1.1 and suppose that the set $\{\mathbf{n}_k\}_{k=1}^r$ of all primitive generators spans \mathbb{R}^m .*

(i) *Then there is a natural isomorphism*

$$(1.4) \quad X_\Sigma \cong \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}, \mathbb{C}^*)/G_\Sigma.$$

(ii) *If $f : \mathbb{CP}^s \rightarrow X_\Sigma$ is a holomorphic map, then there exists an r -tuple $D = (d_1, \dots, d_r) \in (\mathbb{Z}_{\geq 0})^r$ of non-negative integers satisfying the condition $\sum_{k=1}^r d_k \mathbf{n}_k = \mathbf{0}_m$, and homogenous polynomials $f_i \in \mathbb{C}[z_0, \dots, z_s]$ of degree d_i ($i = 1, 2, \dots, r$) such that the polynomials $\{f_i\}_{i \in \sigma}$ have no common root except $\mathbf{0}_{s+1} \in \mathbb{C}^{s+1}$ for each $\sigma \in I(\mathcal{K}_\Sigma)$ and that the diagram*

$$(1.5) \quad \begin{array}{ccc} \mathbb{CP}^{s+1} \setminus \{\mathbf{0}_{s+1}\} & \xrightarrow{(f_1, \dots, f_r)} & \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}, \mathbb{C}^*) \\ \gamma_s \downarrow & & \downarrow q_\Sigma \\ \mathbb{CP}^s & \xrightarrow{f} & \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}, \mathbb{C}^*)/G_\Sigma = X_\Sigma \end{array}$$

¹In this paper by a simplicial complex K we always mean an *abstract simplicial complex*, and we always assume that a simplicial complex K contains the empty set \emptyset .

is commutative, where two map $\gamma_s : \mathbb{C}^{s+1} \setminus \{\mathbf{0}_{s+1}\} \rightarrow \mathbb{CP}^s$ and $q_\Sigma : \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}, \mathbb{C}^*) \rightarrow X_\Sigma$ denote the canonical Hopf fibering and the canonical projection, respectively. In this case, we call this holomorphic map f a holomorphic map of degree $D = (d_1, \dots, d_r)$ and we represent it as $f = [f_1, \dots, f_r]$.

(iii) If $g_i \in \mathbb{C}[z_0, \dots, z_s]$ is a homogenous polynomial of degree d_i ($1 \leq i \leq r$) such that $f = [f_1, \dots, f_r] = [g_1, \dots, g_r]$, there exists some element $(\mu_1, \dots, \mu_r) \in G_\Sigma$ such that $f_i = \mu_i \cdot g_i$ for each $1 \leq i \leq r$. \square

Assumptions. From now on, let Σ be a fan in \mathbb{R}^m as in Definition 1.1, and assume that X_Σ is simply connected.² Thus, we can identify $X_\Sigma = \mathcal{Z}_{\mathcal{K}_\Sigma}(\mathbb{C}, \mathbb{C}^*)/G_\Sigma$, and we shall assume that the following condition holds.

(1.5.1) There is an r -tuple $D = (d_1, \dots, d_r) \in (\mathbb{Z}_{\geq 1})^r$ such that $\sum_{k=1}^r d_k \mathbf{n}_k = \mathbf{0}_m$.

2 Spaces of rational curves on a toric variety

Let P^d denote the space of all monic polynomials $f(z) = z^d + a_1 z^{d-1} + \dots + a_{d-1} z + a_d \in \mathbb{C}[z]$ of degree d , and we set

$$(2.1) \quad P^D = P^{d_1} \times P^{d_2} \times \dots \times P^{d_r}.$$

Now let $D = (d_1, \dots, d_r) \in (\mathbb{Z}_{\geq 1})^r$ be an r -tuple of positive integers satisfying (1.5.1) and consider a base point preserving holomorphic map $f = [f_1, \dots, f_r] : \mathbb{CP}^s \rightarrow X_\Sigma$ of the degree D for the case $s = 1$.

In this situation, we make the identification $\mathbb{CP}^1 = S^2 = \mathbb{C} \cup \infty$ and choose the points ∞ and $[1, 1, \dots, 1]$ as the base points of \mathbb{CP}^1 and X_Σ respectively. Then, by setting $z = \frac{z_0}{z_1}$, for each $1 \leq k \leq r$ we can view f_k as a monic polynomial $f_k(z) \in P^{d_k}$ in the complex variable z . Thus we can identify the space $\text{Hol}_D^*(S^2, X_\Sigma)$ of all base point preserving holomorphic maps $f : S^2 \rightarrow X_\Sigma$ of the degree D with some spaces of r -tuples $(f_1(z), \dots, f_r(z)) \in P^D$ as follows.

Definition 2.1. (1) First consider the case that the r -tuple $D = (d_1, \dots, d_r) \in (\mathbb{Z}_{\geq 1})^r$ satisfies the condition (1.5.1).

In this case, for any r -tuple $D = (d_1, \dots, d_r) \in (\mathbb{Z}_{\geq 1})^r$ satisfying the condition (1.5.1), one can identify the space $\text{Hol}_D^*(S^2, X_\Sigma)$ with the space all r -tuples $(f_1(z), \dots, f_r(z)) \in P^D$ satisfying the condition

- (†) For any $\sigma = \{i_1, \dots, i_s\} \in I(\mathcal{K}_\Sigma)$, the polynomials $f_{i_1}(z), \dots, f_{i_s}(z)$ have no common root i.e. $(f_{i_1}(\alpha), \dots, f_{i_s}(\alpha)) \neq (0, \dots, 0)$ for any $\alpha \in \mathbb{C}$.

²One can show that the set $S = \{\mathbf{n}_k : 1 \leq k \leq r\}$ of primitive generators spans \mathbb{Z}^m over \mathbb{Z} if X_Σ is simply connected. Thus the set S also spans \mathbb{R}^m and the assumption of Theorem 1.2 is satisfied.

Then define the natural inclusion map

$$(2.2) \quad i_D : \text{Hol}_D^*(S^2, X_\Sigma) \rightarrow \text{Map}^*(S^2, X_\Sigma) = \Omega^2 X_\Sigma$$

by

$$(2.3) \quad i_D(f_1(z), \dots, f_r(z))(\alpha) = \begin{cases} [f_1(\alpha), \dots, f_r(\alpha)] & \text{if } \alpha \in \mathbb{C} \\ [1, 1, \dots, 1] & \text{if } \alpha = \infty \end{cases}$$

where we choose the points ∞ and $[1, 1, \dots, 1]$ as the base points of S^2 and X_Σ .

Since $\text{Hol}_D^*(S^2, X_\Sigma)$ is path-connected, the image of i_D is contained in a certain path-component of $\Omega^2 X_\Sigma$, which is denoted by $\Omega_D^2 X_\Sigma$. Thus we have a natural inclusion

$$(2.4) \quad i_D : \text{Hol}_D^*(S^2, X_\Sigma) \rightarrow \text{Map}_D^*(S^2, X_\Sigma) = \Omega_D^2 X_\Sigma.$$

(2) Next we shall consider the general case.

Indeed, for each r -tuple $D = (d_1, \dots, d_r) \in (\mathbb{Z}_{\geq 1})^r$ of positive integers, let H_D^Σ denote the space of r -tuples $(f_1(z), \dots, f_r(z)) \in P^D$ satisfying the condition (\dagger) .

Note that $H_D^\Sigma = \text{Hol}_D^*(S^2, X_\Sigma)$ when the condition $\sum_{k=1}^r d_k \mathbf{n}_k = \mathbf{0}_m$ is satisfied. \square

Definition 2.2. (i) A map $f : X \rightarrow Y$ is called a *homology equivalence through dimension N* (resp. a *homotopy equivalence through dimension N*) if the induced homomorphism $f_* : H_k(X, \mathbb{Z}) \rightarrow H_k(Y, \mathbb{Z})$ (resp. $f_* : \pi_k(X) \rightarrow \pi_k(Y)$) is an isomorphism for any $k \leq N$.

(ii) We say that a set $S = \{\mathbf{n}_{i_1}, \dots, \mathbf{n}_{i_s}\}$ is a *primitive collection* if it does not span a cone in Σ but any proper subset of it does. Then define integers $r_{\min}(\Sigma)$ and $d(D, \Sigma)$ by

$$(2.5) \quad r_{\min}(\Sigma) = \min\{s = \text{card}(S) \in \mathbb{Z}_{\geq 1} : S = \{\mathbf{n}_{i_1}, \dots, \mathbf{n}_{i_s}\} \text{ is a primitive collection}\},$$

$$(2.6) \quad d(D; \Sigma) = (2r_{\min}(\Sigma) - 3)d_{\min} - 2, \text{ where } d_{\min} = \min\{d_1, \dots, d_r\}.$$

Atiyah-Jones-Segal type Theorem. The main purpose of this note is to report the following result.

Theorem 2.3 ([16]). *Let X_Σ be a simply connected non-singular toric variety, and let $D = (d_1, \dots, d_r) \in (\mathbb{Z}_{\geq 1})^r$ be an r -tuple of positive integers such that $\sum_{k=1}^r d_k \mathbf{n}_k = \mathbf{0}_m$.*

Then the inclusion map

$$i_D : \text{Hol}_D^*(S^2, X_\Sigma) \rightarrow \Omega_D^2 X_\Sigma \simeq \Omega^2 \mathcal{Z}_{\mathcal{K}_\Sigma}(D^2, S^1)$$

is a homotopy equivalence through dimension $d(D; \Sigma)$ if $r_{\min}(\Sigma) \geq 3$, and it is a homology equivalence through dimension $d(D; \Sigma) = d_{\min} - 2$ if $r_{\min}(\Sigma) = 2$. \square

Corollary 2.4. *Let X_Σ be a simply connected non-singular toric variety, and suppose that there is an r -tuple $D_* = (m_1, \dots, m_r) \in (\mathbb{Z}_{\geq 1})^r$ of positive integers such that $\sum_{k=1}^r m_k \mathbf{n}_k = \mathbf{0}_m$. Then for each $D = (d_1, \dots, d_r) \in (\mathbb{Z}_{\geq 1})^r$ of positive integers, there is a map*

$$j_D : H_D^\Sigma \rightarrow \Omega_D^2 X_\Sigma \simeq \Omega^2 \mathcal{Z}_{\mathcal{K}_\Sigma}(D^2, S^1)$$

which is a homotopy equivalence through dimension $d(D; \Sigma)$ if $r_{\min}(\Sigma) \geq 3$, and a homology equivalence through dimension $d(D; \Sigma) = d_{\min} - 2$ if $r_{\min}(\Sigma) = 2$. \square

Related topics and problems. Finally we shall comment about the related topics.

Remark 2.5. (i) When $X_\Sigma = \mathbb{CP}^s$ with $s \geq 2$, we can obtain the more sharper result (see the detail in [13] and [14]).

(ii) If X_Σ is compact, M. Guest [7] proved that the map i_D is a homotopy equivalence through dimension $d_{\min} - 1$ and this result is stronger than that of Theorem 2.3 when $r_{\min}(\Sigma) = 2$. In this reason the author and A. Kozłowski are wondering whether the map i_D might be a *homotopy equivalence* through dimension $d(D, \Sigma)$ even if $r_{\min}(\Sigma) = 2$.

(iii) More generally one can consider the similar problem for the inclusion map $i_D : \text{Hol}_D^*(\mathbb{CP}^s, X_\Sigma) \rightarrow \text{Map}_D^*(\mathbb{CP}^s, X_\Sigma)$ when $s \geq 2$, and this problem was really investigated by J. Mostovoy and E. Munguia-Villanueva in [21].

(iv) One can consider the space of resultants related to the space $\text{Hol}_D^*(S^2, X_\Sigma)$ of rational curves on toric varieties and it seems very interesting to investigate whether a similar Atiyah-Jones-Segal conjecture holds for this space. Indeed, when $X_\Sigma = \mathbb{CP}^s$, this problem was solved very nicely in [15]. In the subsequent paper [18], Kozłowski and the author will discuss about the homotopy types of the space of resultants related to toric varieties. \square

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